JOURNAL OF APPROXIMATION THEORY **34**, 264–276 (1982)

Classes of Functions with Explicit Best Uniform Approximations*

HAROLD Z. OLLIN

Massachusetts Institute of Technology, Lincoln Laboratory, Lexington, Massachusetts 02173

AND

IRVING GERST

Department of Applied Mathematics and Statistics, State University of New York, Stony Brook, New York 11794

Communicated by T. J. Rivlin

Received February 2, 1981

1. INTRODUCTION

This paper concerns the construction of forms of the error function, $\varepsilon_n(x) = f(x) - p_n^*(x)$, where p_n^* is the best uniform polynomial approximation of degree n to a continuous function f on $[-1, +1]$. We show that it is always possible and, from the viewpoint of obtaining explicit results, expedient to write the error as $\varepsilon_n = \alpha \cos(n\theta + \phi)$, where $x = \cos\theta$, $|\alpha| = E_n(f)$, the uniform norm of $\varepsilon_n(x)$, and the phase angle ϕ is a continuous function of θ, depending on f and n. Our classes of explicit best approximations arise from a novel method of determining suitable phase angles in this representation of $\varepsilon_n(x)$. The details of this procedure are given in Section 3. It will be evident from the discussion there that our method yields transcendental functions $\varepsilon_n(x)$ as well as rational ones. Although we do not pursue a study of the former systematically in this paper, we will illustrate their occurrence by means of an example (cf. Example 2 of Section 5). The remainder of the paper is devoted to a detailed investigation of various classes of rational $\varepsilon_n(x)$ which are generated by our process.

Bernstein [1] used phase angles to determine the class of rational functions which deviate least from zero on $[-1, +1]$, where the numerator is monic of degree $n + m$, and the denominator is monic of degree m and has

* This work was sponsored by the Department of the Army. The United States Government assumes no responsibility for the information presented.

0021-9045/82/030264–13$02.00/0

arbitrary but fixed zeros outside of $[-1, +1]$. These functions are $\alpha \cos(n\theta + \delta_1 + \cdots + \delta_m)$, where each δ_i is constructed using one of the fixed poles, and α is chosen to render the appropriate terms monic. One of the results of our paper is the construction of elements of this class, and the corresponding phase angles.

In [5] Rivlin exhibited p_n^* for

$$f(x) = \sum_{j=0}^{\infty} t^j T_{aj+b}(x) = \frac{T_b(x) - tT_{|b-a|}(x)}{1 + t^2 - 2tT_a(x)},$$

$a > 0$, $b \geqslant 0$, a and b integers, $-1 < t < +1$, where $T_n(x)$ is the Chebyshev polynomial of the first kind. He showed that

$$\varepsilon_n(\cos\theta) = \frac{t^{k+1}}{1-t^2} \cos[(ak+b)\,\theta + \phi(\theta)]$$

for $ak + b \leqslant n < a(k+1) + b$, where the exhibited ϕ depended on a but was independent of k. It turns out that the phase angle here, whose genesis is not discussed in [5], is one instance of a class of ϕ's obtained in Section 4 of our paper. Thus, our results represent a wide generalization of those given in [5], at least as far as phase angle is concerned.

Also, in connection with [5] and phase angles in general, we note that the first author considered the situations where, for a fixed $f(x)$, ϕ is the same function of θ for all the p_n^*. He showed [4] that this property holds, uniquely, for certain of the functions given by Rivlin in [5].

More precisely, if f is continuous on $[-1, +1]$ and

$$f(\cos\theta) - p_{ak}^*(\cos\theta) = \alpha_k \cos(ak\theta + \phi)$$

for all $k \geqslant 0$ and a fixed positive integer a, where ϕ is a continuous function of θ, independent of k, and α_1, α_2 are nonzero, then f is $\sum_{j=0}^{\infty} t^j T_{aj}$, up to multiplicative and additive constant factors.

Many authors have employed phase angle methods to approximate functions and their best approximations. Prominent among these is Stiefel [7, 8], with whom the nomenclature "phase function" in the context of Chebyshev approximation seems to have originated. He described certain phase functions, which are related to the phase angles used in this paper, and constructed them by an iterative numerical procedure. The structure of error curves involving phase functions like those of Stiefel is discussed further in Rowland [6]. Clenshaw [2] used phase angle methods to approximate best approximations to a polynomial by one of lower degree. That work is expanded and generalized by Darlington in [3].

2. PRELIMINARIES

An essential part of this work involves using the sine and cosine functions with arguments which are functions of some angle. The following lemmas provide some needed results and are given without proof.

LEMMA 1. *Let $u(\theta)$ and $v(\theta)$ be continuous, real functions of θ defined on $\theta_1 \leqslant \theta \leqslant \theta_2$, such that $u^2(\theta) + v^2(\theta) = 1$. Then we can write $u(\theta) = \cos \phi(\theta)$ and $v(\theta) = \sin \phi(\theta)$, where $\phi(\theta)$ is continuous for $\theta_1 \leqslant \theta \leqslant \theta_2$.*

We may note that certain particular values of u and v correspond to special values of ϕ, as indicated in Table I, where k is an integer.

The behavior of ϕ in $[\theta_1, \theta_2]$ can be somewhat described in terms of the appearance of these special values as θ goes from θ_1 to θ_2. Some results are summarized in the following lemma.

LEMMA 2. *Let $u(\theta)$ and $v(\theta)$ be given as in Lemma 1. If either one of $u(\theta)$, $v(\theta)$ has zeros at both endpoints but not in the interior of $[\theta_1, \theta_2]$, while the other function takes each of the values $+1$, -1 at an endpoint, then ϕ maps onto the intervals indicated in Tables II and III, where k is an integer, and $\phi[\theta_1, \theta_2]$ is the range of ϕ.*

We may also note that if we take two consecutive zeros θ_1, θ_2 of either $u(\theta)$ or $v(\theta)$, with the other function taking the same one of the values $+1$,

TABLE I

$u(\theta)$	$v(\theta)$	$\phi(\theta)$
$+1$	0	$2k\pi$
-1	0	$(2k+1)\pi$
0	$+1$	$(2k+\frac{1}{2})\pi$
0	-1	$(2k+\frac{3}{2})\pi$

TABLE II

$u(\theta_1) = u(\theta_2) = 0$

$v(\theta_1)$	$v(\theta_2)$	sgn u in (θ_1, θ_2)	$\phi[\theta_1, \theta_2]$
$+1$	-1	$+1$	$[(2(k+1)+\frac{1}{2})\pi, (2k+\frac{3}{2})\pi]$
$+1$	-1	-1	$[(2k+\frac{1}{2})\pi, (2k+\frac{3}{2})\pi]$
-1	$+1$	$+1$	$[(2k+\frac{3}{2})\pi, (2(k+1)+\frac{1}{2})\pi]$
-1	$+1$	-1	$[(2k+\frac{3}{2})\pi, (2k+\frac{1}{2})\pi]$

TABLE III

$v(\theta_1) = v(\theta_2) = 0$

$u(\theta_1)$	$u(\theta_2)$	sgn v in (θ_1, θ_2)	$\phi[\theta_1, \theta_2]$
+1	−1	+1	$[2k\pi, (2k+1)\pi]$
+1	−1	−1	$[2(k+1)\pi, (2k+1)\pi]$
−1	+1	+1	$[(2k+1)\pi, 2k\pi]$
−1	+1	−1	$[(2k+1)\pi, 2(k+1)\pi]$

-1 at both of these points, then $\phi(\theta_1) = \phi(\theta_2)$, with value in accordance with Table I. The sign between the consecutive zeros determines the quadrant containing $\phi(\theta)$ for $\theta_1 < \theta < \theta_2$.

3. Consideration of Phase Angles

According to the Chebyshev Alternation Theorem, $f(x) - p_n^*(x)$ takes its extreme values $\pm\alpha$, $|\alpha| = E_n(f)$, with alternating signs at least $n+2$ times in $[-1, +1]$. This corresponds to extrema of $\varepsilon_n = f(\cos\theta) - p_n^*(\cos\theta)$ in $0 \leqslant \theta \leqslant \pi$. *We claim that the error ε_n can always be described as $\alpha\cos(n\theta + \phi(\theta))$.* Since $|\varepsilon_n/\alpha| \leqslant 1$, we can choose the function ϕ to describe the behavior of $f - p_n^*$ as a variation in the argument of the cosine function. In particular, we write $\phi(\theta) = \cos^{-1}(\varepsilon_n/\alpha) - n\theta$ and call ϕ the phase of the error. The choice of ϕ depends on f and n. By choosing $\phi(0)$ to be in $[0, 2\pi)$, and considering $e^{i\phi(\theta)}$ to be on the Riemann surface corresponding to e^z, we see that $\phi(\theta)$ varies continuously as θ goes from 0 to π. The argument $n\theta + \phi$ will have a range including the closed interval with endpoints at $\phi(0)$ and $n\pi + \phi(\pi)$. The range must provide as many extrema of cosine with alternating signs as required for $f - p_n^*$. The smoothness of ϕ will depend directly on the smoothness of f.

Conversely, given the function $\cos(n\theta + \phi)$, with ϕ a continuous function of θ in $[0, \pi]$, a *sufficient condition* to guarantee that $\varepsilon_n = \alpha\cos(n\theta + \phi)$ is the error for a best uniform polynomial approximation of degree n is that the range of $(n\theta + \phi)$ includes at least $n+2$ points at which cosine takes its extreme values with alternating changes in sign. This can be proven by the method employed in Theorem 4. Since $n\theta$ increases linearly from 0 to $n\pi$, it is sufficient to have $\phi(0)$ and $\phi(\pi)$ be multiplies of π, with $\phi(\pi) - \phi(0) \geqslant \pi$.

In order to satisfy the preceding sufficient condition, it would be desirable to have a method of selecting ϕ which would give us some information about its range a priori. It was this objective that led us to the procedure described next. It will become evident in Section 4 how this selection process relates to the range of ϕ.

Since $\varepsilon_n/\alpha = \cos n\theta \cos \phi - \sin n\theta \sin \phi$, it suffices to determine $\cos \phi$ and $\sin \phi$ as functions of θ. Lemma 1 then suggests the following. Let $w(z)$ be a function analytic in a region including the unit circle, which maps the unit circle into itself, that is, $|w(e^{i\theta})| = 1$ for $0 \leqslant \theta \leqslant 2\pi$. If $w(e^{i\theta}) = u(\theta) + iv(\theta)$, u and v real functions, then we can take $\cos \phi = u(\theta)$, $\sin \phi = v(\theta)$. The type of error function that we get here will correspond, in a rough way, to the type of function $w(z)$ with which we started. However, it is rather difficult to recognize when a function $w(z)$ has the required mapping property. The use of a well-known transformation from analytic function theory will overcome this difficulty.

Consider the bilinear transformation $z = (1 - p)/(1 + p)$, and its inverse $p = (1 - z)/(1 + z)$. These transformations map the imaginary axis and the unit circle onto each other. Also consider the function $f(p)$ which is connected to $w(z)$ by the following relations:

$$f(p) = \frac{1 - w((1-p)/(1+p))}{1 + w((1-p)/(1+p))}, \tag{1}$$

$$w(z) = \frac{1 - f((1-z)/(1+z))}{1 + f((1-z)/(1+z))}. \tag{2}$$

In (1), if w maps the unit circle into itself, then f maps the imaginary axis into itself, and conversely in (2). Given either w or f with the above property, we can find the other via (1) or (2).

THEOREM 1a. *Assume the function $f(p)$ is analytic, except for possible poles, in a region including the imaginary axis. Also, assume $f(p)$ is a real function, that is, $f(\bar{p}) = \overline{f(p)}$. Then a necessary and sufficient condition that, for each real t, $f(it) = i\alpha$, α real (and dependent on t), is that $f(p)$ is an odd function.*

Proof. Sufficiency follows from the fact that $f(it) = a + bi$ implies $f(-it) = a - bi = -a - bi$, so $a = 0$. Necessity follows from the fact that $f(it) = i\alpha$ implies $f(-it) = -i\alpha = -f(it)$, and so by the principle of the permanence of functional relations, $f(-p) = -f(p)$. Q.E.D.

Note that we can extend this class by allowing essential singularities on the imaginary axis, say, at $p = it_0$, as long as $\lim_{t \to t_0} f(it)$, t real, exists (including the case when it is infinite). To sum up, if $f(p)$ is real and odd, then the corresponding $w(z)$ will have the required mapping property. The following corollary is now obvious.

COROLLARY 1. *Let $f(p)$ be an odd function satisfying the conditions of Theorem* 1a. *Then $w(z)$, given via Eq.* (2), *maps the unit circle into itself, and $w(e^{i\theta}) = u(\theta) + iv(\theta)$ satisfies $u^2(\theta) + v^2(\theta) = 1$.*

By Lemma 1 we can write $u(\theta) = \cos\phi(\theta)$ and $v(\theta) = \sin\phi(\theta)$, so $e^{i\phi(\theta)} = w(e^{i\theta})$. It is this function ϕ upon which our considerations are focused. The question arises as to what restrictions on f will give ϕ, via the steps taken above, which will map $[0, \pi]$ onto $[k_1\pi, k_2\pi]$, where k_1 and k_2 are both integers. Such a ϕ will exhibit the behavior of a phase angle of a special type. We now give a useful formulation which permits us to calculate $\cos\phi$ and $\sin\phi$ directly from $f(p)$.

Letting $z = e^{i\theta}$, we observe that $p = (1 - z)/(1 + z) = -i\tan(\theta/2)$. Write this as $p = -is$, $s = \tan(\theta/2)$, and define the real function F by $f(it) = iF(t)$. We have

$$\begin{aligned} w(e^{i\theta}) &= \frac{1 - f(-is)}{1 + f(-is)} = \frac{1 + iF(s)}{1 - iF(s)} \\ &= \frac{1 - F^2(s)}{1 + F^2(s)} + i\frac{2F(s)}{1 + F^2(s)} \\ &= \cos\phi(\theta) + i\sin\phi(\theta). \end{aligned}$$

The next lemma shows a relation between the behavior of ϕ and f.

LEMMA 3. *Let $f(p)$ be an odd function satisfying the conditions of Theorem* 1a *and let $w(z)$ be given by* (2). *Then $w(e^{i\theta})$ will attain the values $+1$ or -1 if and only if f has a zero or pole, respectively, at the corresponding point on the imaginary axis.*

Proof. $w(e^{i\theta})$ is real for some θ if and only if $w((1-p)/(1+p))$ is real for the corresponding p on the imaginary axis. Then by the inverse mapping (1), $f(p)$ is real for that value of p. Since $f(it) = iF(t)$, with $F(t)$ real, $f(p)$ must be zero or infinity. Equation (1) shows $f(p) = 0$ corresponds to $w((1-p)/(1+p)) = +1$, and $f(p) = \infty$ corresponds to $w((1-p)/(1+p)) = -1$. Q.E.D.

If $u(\theta) = \pm 1$, then $v(\theta) = 0$. The function $v(\theta)$ will change sign in passing through a zero if and only if the corresponding zero or pole of f on the imaginary axis is of odd multiplicity. This can be seen as follows.

Again note the fact that $w(e^{i\theta})$ equals $w((1-p)/(1+p))$ for $p = -i\tan(\theta/2)$ and substitute $w(e^{i\theta}) = u(\theta) + iv(\theta)$ into (1). Then

$$f(p) = \frac{1 - w(e^{i\theta})}{1 + w(e^{i\theta})} = \frac{-v(\theta)\,i}{1 + u(\theta)}.$$

Therefore, for the θ and p which correspond to each other, $\operatorname{sgn} v(\theta) =$

$-\operatorname{sgn}\operatorname{Im} f(p)$ whenever $v(\theta) \neq 0$, and $v(\theta)$ will change sign exactly when $\operatorname{Im} f(p)$ does so. $\operatorname{Im} f(p)$ changes sign as p varies along the imaginary axis when p goes through a zero or pole of odd multiplicity, as we can verify by considering the series expansion for f taken about such a zero or pole. For example, if $p_0 = it_0$ is a zero of order $2k+1$, with t_0 real, then $f(p) = c(p - it_0)^{2k+1} +$ (higher order terms), $c \neq 0$, is the Taylor expansion about the point it_0. Taking $p = it$, t real, we get

$$\begin{aligned} f(it) &= c(it - it_0)^{2k+1} + \cdots \\ &= \pm ic(t - t_0)^{2k+1} + \cdots . \end{aligned}$$

Then $\operatorname{Im} f(it) = F(t) = \pm c(t - t_0)^{2k+1} + \cdots$. This implies that c is real and that $\operatorname{Im} f$ changes sign as t goes through t_0.

As will be seen in the next section, there is an intimate connection between the range of ϕ and the location of the poles and zeros of $f(p)$ on the imaginary axis. The specifics depend on a detailed analysis which we now proceed to carry out when $f(p)$ is a rational function.

4. Phase Angles Generated from Rational Functions

We begin by giving a variation of the previous theorem relating specifically to the case where $f(p)$ is a rational function.

Theorem 1b. *Let f be a rational function with real coefficients. Then f maps the imaginary axis into itself if and only if all the powers in either one of the numerator and denominator are odd, while all the powers in the other are even.*

Proof. Sufficiency being clear, we prove only necessity. Since $f(p)$ is odd, the result is immediate if $f(p)$ or $1/f(p)$ is a polynomial. Hence, let the unique product representation of $f(p)$ be

$$f(p) = \frac{Ap^{n_0} \prod_{i=1}^{n} (p - a_i)^{s_i}}{\prod_{j=1}^{n} (p - b_j)^{t_j}},$$

where the a_i, b_j are non-zero, distinct complex numbers, s_i, t_j are positive integers, A is a constant, and n_0 is an integer. Comparing the factors on both sides of the equality $f(p) = -f(-p)$, we see that for each factor $p - a_i$ in the numerator there is also a factor $p + a_i$ and with the same multiplicity, giving the even product $(p^2 - a_i^2)^{s_i}$. Similarly for the denominator. Comparing $f(-1)$ with $-f(1)$, we now have $(-1)^{n_0} = -1$, so that n_0 is odd. Q.E.D.

Take f to be in the form

$$f(x) = A(x^2)/xD(x^2) \tag{3a}$$

or

$$f(x) = xB(x^2)/C(x^2) \tag{3b}$$

and allow no common factors in the numerator and denominator. We then form $w(z)$ as in Eq. (2) and employ Lemma 3 and the succeeding discussion to determine functional forms for f which give the desired behavior of ϕ. Observe that θ varies from 0 to π as p varies from 0 to ∞ along the negative imaginary axis.

Let Z and P indicate the degrees of the numerator and denominator of a rational function. Let I^- denote negative imaginary axis, plus zero and infinity. Let Z^- and P^- indicate the number of zeros and poles (without regard to each one's multiplicity) on I^-.

THEOREM 2. *Let $f(x) = xB(x^2)/C(x^2)$ with $Z > P$. Let all the zeros and poles on I^- be of odd multiplicity and interlacing. Also, let the imaginary part of $f(p)$ be negative in the open interval between the origin and the first negative imaginary pole. Then $\phi(\theta)$, formed as before, varies from 0 to $[2(Z^-) - 1]\pi$ as θ varies from 0 to π.*

Proof. We note that $Z^- = P^-$. Also, $f(0) = 0$ and $f(\infty) = \infty$ are implied by the form of f and the inequality $Z > P$. [This zero and pole are of odd multiplicity because of the opposite parities of the degrees of $xB(x^2)$ and $C(x^2)$.] Write $r + 1 = Z^- = P^-$. We can denote the zeros and poles on I^- by $0, -z_1 i, -z_2 i, \ldots, -z_r i$ and $-p_1 i, -p_2 i, \ldots, -p_r i, \infty$, respectively, where $0 < p_1 < z_1 < \cdots < p_r < z_r < \infty$. By Lemma 3, $v(\theta)$ is zero and $u(\theta)$ alternates from $+1$ to -1 at the values of θ which correspond to these points. Denote these values by $0 = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_{2r} < \theta_\infty = \pi$. Without loss of generality, we may set $\phi(0) = 0$. Since $\operatorname{Im} f(p) < 0$ in $(0, -p_1 i)$, $v(\theta)$ is positive in $(0, \theta_1)$. By Lemma 2, ϕ varies from 0 to π as θ goes from 0 to θ_1. Because of the odd multiplicity of $-p_1 i$, $v(\theta)$ changes sign as it passes through θ_1, and by Lemma 2, ϕ varies from π to 2π as θ goes from θ_1 to θ_2. Repeating the argument we find that ϕ varies through an additional length π as θ goes from θ_j to θ_{j+1}, until θ reaches π. In total ϕ has varied from 0 to $(2r + 1)\pi$. Q.E.D.

THEOREM 3. *Let $f(x) = xB(x^2)/C(x^2)$ with $Z < P$. Let all the zeros and poles on I^- be of odd multiplicity and interlacing. Let $\operatorname{Im} f(p) < 0$ in the open interval between the origin and the first negative imaginary pole. Then $\phi(\theta)$ varies from 0 to $2[(Z^-) - 1]\pi$ as θ varies from 0 to π.*

Proof. Here $Z^- = P^- + 1$, and $f(0) = f(\infty) = 0$. With $r + 1 = Z^-$, we write the zeros and poles as 0, $-z_1 i, -z_2 i, \ldots, -z_{r-1} i, \infty$ and $-p_1 i$, $-p_2 i, \ldots, -p_r i$, respectively, where $0 < p_1 < z_1 < \cdots < z_{r-1} < p_r < \infty$. The corresponding values of θ are $0 = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_{2r-1} < \theta_\infty = \pi$. By the same argument as that in the previous proof, $\phi(0) = 0$ and ϕ varies a length π as θ varies from θ_j to θ_{j+1}. In total ϕ varies from 0 to $2r\pi$. Q.E.D.

These last two theorems give ϕ's which vary from 0 to either odd or even positive multiples of π. If we had used $f(x) = A(x^2)/xD(x^2)$, the results would be similar. We observe that

$$\frac{1 - A(x^2)/xD(x^2)}{1 + A(x^2)/xD(x^2)} = -\frac{1 - xD(x^2)/A(x^2)}{1 + xD(x^2)/A(x^2)}.$$

Therefore, the $w(z)$ arising from $A(x^2)/xD(x^2)$ is simply the negative of the $w(z)$ which comes from the reciprocal. The $u(\theta)$ and $v(\theta)$ also change sign. This means the function $\phi(\theta)$ is a length π out of phase with the ϕ coming from the reciprocal. The following corollaries are now obvious.

COROLLARY 2. *Let $f(x) = A(x^2)/xD(x^2)$ with $Z < P$. Let the zeros and poles on I^- be of odd multiplicity and interlacing. Let* $\operatorname{Im} f(p) > 0$ *in the open interval between the origin and the first negative imaginary zero. Then $\phi(\theta)$ varies from π to $2(Z^-)\pi$ as θ varies from 0 to π.*

COROLLARY 3. *Let $f(x) = A(x^2)/xD(x^2)$ with $Z > P$. Let the zeros and poles on I^- be of odd multiplicity and interlacing. Let* $\operatorname{Im} f(p) > 0$ *in the open interval between the origin and the first negative imaginary zero. Then $\phi(\theta)$ varies from π to $(2Z^- + 1)\pi$ as θ varies from 0 to π.*

We note that $f(0) = \infty$, and $f(\infty) = 0$ in the former corollary, but $f(\infty) = \infty$ in the latter. In both we may choose $\phi(0) = \pi$, and then ϕ varies a length π between each pair of consecutive zeros and poles.

These results can all be extended by permitting ϕ to vary about its initial value before advancing to the next multiple of π. In view of the remarks at the end of Section 2, we see that for $f(x) = xB(x^2)/C(x^2)$, if we insert additional zeros (of any multiplicity) between the origin and the first pole on I^-, without violating the relation between Z and P, and if $\operatorname{Im} f(p) < 0$ between the last such zero and that pole, then ϕ will cover the same length interval as before. Equivalently, for $f(x) = A(x^2)/xD(x^2)$, if we insert additional poles (of any multiplicity) between the origin and the first zero on I^-, without violating that same relation, and if $\operatorname{Im} f(p) > 0$ between the last such pole and that zero, then ϕ covers the same length interval as before.

5. Explicit Approximation Results

If we take ϕ generated by a rational function satisfying a set of conditions given by any of the theorems or corollaries of the previous section, let us assume that a continuous function f satisfies $f(\cos\theta)-p_n^*(\cos\theta)=\alpha_n\cos(n\theta+\phi)$ for some value, or set of values, of n. We can show $\cos n\theta\cos\phi$ is a rational function in $\cos\theta$, with $Z=M+n+1$ and $P=M+1$, where M is the largest even power occurring in either the numerator or denominator of the rational function which generates ϕ. Similarly, $\sin n\theta\sin\phi$ is a rational function with the same denominator and, $Z=M+n+1$ and $P=M+1$. Then, allowing for cancellation of terms, we have that $f-p_n^*$ is a rational function with $Z\leqslant M+n+1$ and $P\leqslant M+1$, since $f(\cos\theta)-p_n^*(\cos\theta)=\alpha_n(\cos n\theta\cos\phi-\sin n\theta\sin\phi)$. The only functions which may satisfy this are rational f's with the same bounds on Z and P. We have the following theorems as a result of this functional form.

THEOREM 4. *Generate ϕ, via Eq.* (2), *from a rational function satisfying the conditions of Theorem* 2 (*or* 3). *Let $m=n+2Z^- -2$ (or $n+2Z^- -3$), where n is a non-negative integer. Let $g(\theta;n,\phi,p,\alpha)=\alpha\cos(n\theta+\phi)+p(\cos\theta)$, where p is a polynomial of degree $k\leqslant m$, and α is real. Then p is the best uniform polynomial approximation of degrees $k, k+1,..., m$, to g on $0\leqslant\theta\leqslant\pi$, with error $|\alpha|$.*

Proof. The function g is rational, and $g-p=\alpha\cos(n\theta+\phi)$. The argument $(n\theta+\phi)$ varies from 0 to $(n+2Z^- -1)\pi$ (or $[n+2(Z^- -1)]\pi$) as θ varies from 0 to π. Therefore, $\cos(n\theta+\phi)$ has $n+2Z^-$ (or $n+2Z^- -1$) alternation points. With our restriction on k, the Chebyshev Alternation Theorem yields the desired result. Q.E.D.

THEOREM 5. *Let ϕ and m be as in the previous theorem. Then $\cos(n\theta+\phi)=r(\cos\theta)-p(\cos\theta)$, where r is a rational function whose numerator has degree no greater than that of the denominator, p is a polynomial of degree $k\leqslant n$, and p is the best uniform polynomial approximation of degrees $k, k+1,..., m$, to r on $0\leqslant\theta\leqslant\pi$.*

Proof. We have seen that $\cos(n\theta+\phi)$ is a rational function with $Z\leqslant M+n+1$ and $P\leqslant M+1$. By eliminating common factors and dividing, we get the indicated form for $\cos(n\theta+\phi)$. The method of the preceding proof then gives the desired result. Q.E.D.

In Example 1 we establish that the result of [5] can be derived as a special case of our results. Finally, in Example 2 we give an example, arising from our method, of an error function which is transcendental.

EXAMPLE 1. We consider the following function for $-1 < t < 1$,

$$f(p) = \frac{(1+t)[(1+p)^a - (1-p)^a]}{(1-t)[(1+p)^a + (1-p)^a]}.$$

This is of the form (3b) for both odd and even values of a. If a is odd, then $Z = a$, $P = a - 1$, $Z^- = P^- = (a+1)/2$. If a is even, then $Z = a - 1$, $P = a$, $Z^- = P^- + 1 = (a+2)/2$. The zeros are at $(1-p)/(1+p) = e^{2\pi k i/a}$, $k = 0, 1, \ldots, a-1$, with values $k = 0, 1, \ldots, [(a-1)/2]$ corresponding to values of p in $[0, -\infty)$, or $0 \leqslant \theta < \pi$. ($[x]$ = greatest integer less than or equal to x.) The poles are at $(1-p)/(1+p) = e^{(2j+1)\pi i/a}$, $j = 0, 1, \ldots, a-1$, with the values $j = 0, 1, \ldots, [(a-2)/2]$ corresponding to values of p in $[0, -\infty)$, or $0 \leqslant \theta < \pi$. For $a =$ odd, $j = (a-1)/2$ gives a pole at $p = -\infty$ $(\theta = \pi)$. This last pole is not counted in P. For $a =$ even, $k = a/2$ gives a zero at $p = -\infty$ $(\theta = \pi)$, which is not counted in Z. The zeros and poles are of multiplicity one, and they interlace. The sign of $\operatorname{Im} f(p)$ between the origin and the first negative imaginary pole $[(1-p)/(1+p) = \exp(\pi/a)\, i]$ can be evaluated at $((1-p)/(1+p))^a = \exp(\pi i/2) = i$. At that point $f(p) = -i((1+t)/(1-t))$, and so $\operatorname{Im} f(p) < 0$ between the origin and the first negative imaginary pole. By Theorems 2 and 3, we know that when we transform f into $w(z)$ and take $w(e^{i\theta}) = e^{i\phi(\theta)}$, ϕ will increase from 0 to $a\pi$ as θ goes from 0 to π.

Using (2), we get

$$w(z) = (z^a - t)^2/(-tz^{2a} + (1+t^2)\, z^a - t).$$

Letting $z = e^{i\theta}$, then separating $w(e^{i\theta})$ into its real and imaginary parts, we find $w(e^{i\theta}) = \cos\phi(\theta) + i \sin\phi(\theta)$, where $\cos\phi$ and $\sin\phi$ are given by

$$\sin\phi = \frac{(1-t^2)\sin a\theta}{1+t^2-2t\cos a\theta},$$

$$\cos\phi = \frac{-2t + (1+t^2)\cos a\theta}{1+t^2-2t\cos a\theta}.$$

Let $n = ak$ and form $\alpha\cos(ak\theta + \phi)$, with $\alpha = t^{k+1}/(1-t^2)$. Then,

$$\alpha\cos(ak\theta+\phi) = \frac{\alpha[\cos a(k+1)\,\theta - 2t\cos ak\theta + t^2\cos a(k-1)\,\theta]}{1+t^2-2t\cos a\theta}.$$

We can verify that

$$\begin{aligned}\alpha\cos(ak\theta+\phi) &= \frac{1-t\cos a\theta}{1+t^2-2t\cos a\theta} - \sum_{j=0}^{k} t^j\cos aj\theta - \frac{t^{k+2}}{1-t^2}\cos ak\theta\\ &= r(\cos\theta) - p(\cos\theta).\end{aligned}$$

In accordance with Theorem 5, the polynomial p, which is of degree $ak = n$, is the best uniform polynomial approximation of degrees ak, $ak + 1,\ldots, m = a(k + 1) - 1$, to the rational function r.

This is the result of Rivlin [5], and is indeed a special case since ϕ is independent of the choice of k.

EXAMPLE 2. Let $f(p) = p \exp((1 + p^2)/(1 - p^2))$ and note that f is analytic on the imaginary axis except for a pole at infinity. Also, $f(0) = 0$. Defining F by $f(it) = iF(t)$, we have $F(t) = t \exp((1 - t^2)/(1 + t^2))$. Therefore, as in Section 3,

$$\cos \phi(\theta) = \frac{1 - s^2 \exp\left[2\left(\frac{1 - s^2}{1 + s^2}\right)\right]}{1 + s^2 \exp\left[2\left(\frac{1 - s^2}{1 + s^2}\right)\right]} = -1 + \frac{2}{1 + s^2 \exp\left[2\left(\frac{1 - s^2}{1 + s^2}\right)\right]},$$

$$\sin \phi(\theta) = \frac{2s \exp\left(\frac{1 - s^2}{1 + s^2}\right)}{1 + s^2 \exp\left[2\left(\frac{1 - s^2}{1 + s^2}\right)\right]},$$

where $s = \tan(\theta/2)$. (Since $f(-i) = -i$, we have conditions analogous to Theorem 2.)

As θ goes from 0 to π, s increases monotonically from 0 to ∞. $\cos \phi$ decreases from $+1$ to -1, while $\sin \phi > 0$ except at the endpoints of the interval. Therefore, ϕ ranges from 0 to π, and $\varepsilon_n = a \cos(n\theta + \phi)$ has $n + 2$ alternations. As in Theorem 4, any polynomial p of degree $k \leqslant n$ is the best uniform polynomial approximation to $\varepsilon_n + p$ for degrees $k, k + 1,\ldots, n$.

Since $s^2 = (1 - \cos \theta)/(1 + \cos \theta) = (1 - x)/(1 + x)$, we have $x = (1 - s^2)/(1 + s^2)$. Also, $s \sin \theta = 1 - \cos \theta = 1 - x$. Hence,

$$\frac{\varepsilon_n}{\alpha} = \cos n\theta \left[-1 + \frac{2(1 + x)}{1 + x + (1 - x)e^{2x}}\right] - \frac{\sin n\theta}{\sin \theta} \frac{2(1 - x^2)e^x}{1 + x + (1 - x)e^{2x}}.$$

Here, $\cos n\theta = T_n(x)$ and $(\sin n\theta)/(\sin \theta) = U_{n-1}(x)$ are the Chebyshev polynomials. Therefore, for example, T_n is the best approximation p_n^* to

$$f(x) = 2(1 + x)\left[\frac{T_n(x) - (1 - x)e^x U_{n-1}(x)}{1 + x + (1 - x)e^{2x}}\right].$$

References

1. S. Bernstein, "Leçons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variable réelle," Gauthier–Villars, Paris, 1926; reprint, Cheisea, New York, 1970.
2. C. W. Clenshaw, A comparison of "best" polynomial approximations with truncated Chebyshev series expansions, *SIAM J. Numer. Anal.* **1** (1964), 26–37.
3. S. Darlington, Analytical approximations to approximations in the Chebyshev sense, *Bell System Tech. J.* **49** (1970), 1–32.
4. H. Z. Ollin, Best uniform polynomial approximation to certain rational functions, *J. Approx. Theory* **4** (1979), 382–392.
5. T. J. Rivlin, Polynomials of best uniform approximation to certain rational functions, *Numer. Math.* **4** (1962), 345–349.
6. J. H. Rowland, On the structure of the error curve in Chebyshev approximation by polynomials, *SIAM J. Numer. Anal.* **4** (1967), 398–405.
7. E. L. Stiefel, Methods—old and new—for solving the Tchebycheff approximation problem, *SIAM J. Numer. Anal.* **1** (1964), 164–176.
8. E. L. Stiefel, Phase methods for polynomial approximation, *in* "Approximation of Functions" (H. L. Garabedian, Ed.), pp. 68–82, Elsevier, Amsterdam, 1965.